Please check the examination details below before entering your candidate information

Candidate surname	Other names

Centre Number | | | | | | Candidate Number | | | | |

Pearson Edexcel Level 3 GCE

Wednesday 13 October 2021 – Afternoon

Time 2 hours

Paper reference **9MA0/02**

Mathematics

Advanced
PAPER 2: Pure Mathematics 2

You must have:
Mathematical Formulae and Statistical Tables (Green), calculator

Total Marks

Candidates may use any calculator allowed by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need*.
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 15 questions in this question paper. The total mark for this paper is 100.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question*.

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.

Turn over

P 6 8 7 3 2 A 0 1 4 8

Pearson

1. In an arithmetic series

- the first term is 16
- the 21st term is 24

(a) Find the common difference of the series.

(2)

(b) Hence find the sum of the first 500 terms of the series.

(2)

Each term in arithmetic sequence has form:

$t_n = a + (n-1)d$

number in sequence   first term   common difference

SO, (a) $16 = a$ because 16 is the first term

$t_{21} = 16 + (21-1)d = 24$

$16 + 20d = 24$

-16

$20d = 8$

$d = \frac{8}{20} = \frac{2}{5}$

(b) Sum of numbers in a sequence:

$S_n = \frac{n}{2}(2a + (n-1)d)$

SO, $S_{500} = \frac{500}{2}\left(2 \times 16 + (500-1) \times \frac{2}{5}\right)$

$= 250\left(32 + 499 \times \frac{2}{5}\right)$

$= 250\left(32 + \frac{998}{5}\right)$

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Question 1 continued

$= 250 \times \frac{1158}{5}$

$= 57900.$

(Total for Question 1 is 4 marks)

2. The functions f and g are defined by

$$f(x) = 7 - 2x^2 \qquad x \in \mathbb{R}$$

$$g(x) = \frac{3x}{5x-1} \qquad x \in \mathbb{R} \qquad x \neq \frac{1}{5}$$

(a) State the range of f

(1)

(b) Find gf(1.8)

(2)

(c) Find $g^{-1}(x)$

(2)

(a) The range is the values of f(x) for every single x value.

$f(x) = 7 - 2x^2$

$x \in \mathbb{R}$ so $x^2 \geqslant 0$ for all x- values

so the smallest value x^2 can take is 0, so the biggest value $f(x) = 7 - 2x^2$ can take is $7 - 0 = 7$

so $f(x) \leqslant 7$

There is no upper limit to x^2 so there is no smallest value for $f(x) = 7 - 2x^2$

range of f(x) : $-\infty < f(x) \leqslant 7$

Remember: Don't use $\leqslant$, $\geqslant$ with $\pm\infty$.

$f(0) = 7$ so use less than and equal to

Question 2 continued

(b) remember $gf(1.8) = g(f(1.8))$ so find $f(1.8)$ first.

$f(1.8) = 7 - 2 \times 1.8^2$

$= 7 - 2 \times 3.24$

$= 7 - 6.48 = 0.52$

Then, put $f(1.8)$ in $g(x)$.

so $gf(1.8) = g(0.52)$

$= \dfrac{3 \times 0.52}{5 \times 0.52 - 1}$

$= 0.975$

(c) $g^{-1}(x)$ is the function that undoes $g(x)$.

If $y = g^{-1}(x)$

$g(y) = gg^{-1}(x) = x$

so let $x = g(y)$

$x = \dfrac{3y}{5y-1}$ sub y into $g(x)$ function

$5xy - x = 3y$ multiply to get rid of the fraction

$5xy - 3y = x$ rearrange so all ys on same side

(Total for Question 2 is 5 marks)

P 6 8 7 3 2 A 0 5 4 8

Q2c cont.

$$y(5x-3) = x$$ take a factor of y out

$$y = \frac{x}{5x-3}$$ divide through to get y on its own

so $$g^{-1}(x) = \frac{x}{5x-3}$$ Remember $y = g^{-1}(x)$ so rewrite to get $g^{-1}(x)$.

3. Using the laws of logarithms, solve the equation

$$\log_3 (12y + 5) - \log_3 (1 - 3y) = 2$$

(3)

$\log_3 (12y + 5) - \log_3 (1 - 3y) = 2$

law of logs: $\log x - \log y = \log \frac{x}{y}$

$\log_3 \left(\frac{12y + 5}{1 - 3y} \right) = 2$ Rearrange into one log.

$\cancel{3}^{\cancel{\log_3} \left(\frac{12y + 5}{1 - 3y} \right)} = 3^2$ make both sides powers of 3

law of logs: $x^{\log_x (y)} = y$

$\frac{12y + 5}{1 - 3y} = 9$ cancel using log laws and evaluate 3^2.

$12y + 5 = 9 - 27y$

$39y = 4$ Rearrange to find y.

$y = \frac{4}{39}$

(Remember: you can check your solution by substituting back into the equation:

$\log_3 \left(12 \times \frac{4}{39} + 5\right) - \log \left(1 - 3 \times \frac{4}{39}\right) = 2$)

4. Given that θ is small and measured in radians, use the small angle approximations to show that

$$4\sin\frac{\theta}{2} + 3\cos^2\theta \approx a + b\theta + c\theta^2$$

where a, b and c are integers to be found.

(3)

$4\sin\frac{\theta}{2} + 3\cos^2\theta$

$\cos^2 x = 1 - \sin^2 x$

$= 4\sin\frac{\theta}{2} + 3(1-\sin^2\theta)$

$= 4\sin\frac{\theta}{2} + 3 - 3\sin^2\theta$

small angle approximations:
$\sin x \approx x.$

so substitute into eqn.

$\approx 4\left(\frac{\theta}{2}\right) + 3 - 3(\theta)^2$

$\approx 2\theta + 3 - 3\theta^2$

$\approx 3 + 2\theta - 3\theta^2$

rearrange for required form.

$a = 3,\ b = 2,\ c = -3$

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5. The curve C has equation

$$y = 5x^4 - 24x^3 + 42x^2 - 32x + 11 \qquad x \in \mathbb{R}$$

(a) Find

(i) $\dfrac{dy}{dx}$

(ii) $\dfrac{d^2y}{dx^2}$

(3)

(b) (i) Verify that C has a stationary point at $x = 1$

(ii) Show that this stationary point is a point of inflection, giving reasons for your answer.

(4)

(a) Differentiating y with respect to x:

(i) $\dfrac{dy}{dx} = 4 \times 5x^3 - 3 \times 24x^2 + 2 \times 42x - 32$

$= 20x^3 - 72x^2 + 84x - 32$

(ii) $\dfrac{d^2y}{dx^2} = 3 \times 20x^2 - 2 \times 72x + 84$

$= 60x^2 - 144x + 84$

(b) Stationary points occur when $\dfrac{dy}{dx} = 0$,

Method 1: Solve $\dfrac{dy}{dx} = 0$ and show $x = 1$ is a solution

$20x^3 - 72x^2 + 84x - 32 = 0$ use calculator to solve.

$\Rightarrow x = \dfrac{8}{5}$, $\underline{x = 1}$

so C has a stationary point at $x = 1$.

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Question 5 continued

method 2: substitute $x = 1$ into $\frac{dy}{dx}$ and show that equals 0.

$$\left.\frac{dy}{dx}\right|_{x=1} = 20 \times 1^3 - 72 \times 1^2 + 84 \times 1 - 32$$

$$= 20 - 72 + 84 - 32$$

$$= 104 - 104$$

$$= 0,$$

so $x=1$ is a stationary point of C.

(b) (ii) Points of inflection occur when $\frac{d^2y}{dx^2} = 0$

method 1: solve $\frac{d^2y}{dx^2} = 0$ and show $x=1$ is solution.

$$\frac{d^2y}{dx^2} = 60x^2 - 144x + 84 = 0$$ solve on calculator

$$\Rightarrow x = \frac{7}{5}, \quad \underline{x=1}$$

so C has point of inflection at $x=1$.

method 2: sub $x=1$ into $\frac{d^2y}{dx^2}$ and show it is 0.

(Total for Question 5 is 7 marks)

Q5 (b) (ii) cont.

$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = 60 \times 1^2 - 144 \times 1 + 84$$

$$= 60 - 144 + 84$$

$$= 144 - 144$$

$$= 0$$

so, c has a point of inflection at $x=1$.

TIP: for (b)(i) and (b)(ii), you can check these on a graphical calculator but shouldn't rely on this as your answer.

6.

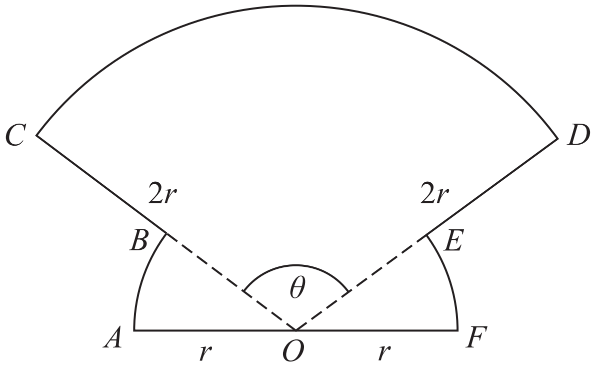


Figure 1

The shape *OABCDEFO* shown in Figure 1 is a design for a logo.

In the design

- *OAB* is a sector of a circle centre *O* and radius *r*
- sector *OFE* is congruent to sector *OAB*
- *ODC* is a sector of a circle centre *O* and radius $2r$
- *AOF* is a straight line

Given that the size of angle *COD* is θ radians,

(a) write down, in terms of θ, the size of angle *AOB*

(1)

(b) Show that the area of the logo is

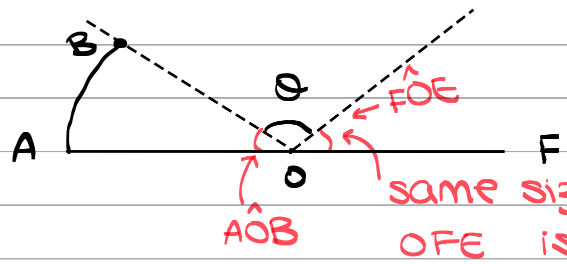
$$\frac{1}{2}r^2(3\theta + \pi)$$

(2)

(c) Find the perimeter of the logo, giving your answer in simplest form in terms of r, θ and π.

(2)

(a)



same size as AÔB because sector OFE is congruent to sector OAB.

so, $\pi = \theta + 2A\hat{O}B$ $\quad$ π is angle of straight line in radians

$\Rightarrow A\hat{O}B = \dfrac{\pi - \theta}{2}$ $\quad$ Rearrange for AOB

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Question 6 continued

(b) Area of logo = area of ODC + area of OAB + area of OFE.

since OAB and OFE are congruent, they have the same area.

so area of logo = area of ODC + 2x area of OAB.

Remember:

angle radius

area of a sector = $\frac{\phi}{2\pi} \times \pi R^2 = \frac{\phi R^2}{2}$

so area of ODC = $\frac{\theta}{2} \times (2r)^2$

$= \frac{\theta}{2} \times 4r^2$ Apply formula with $\phi = \theta$, $R = 2r$

$= 2\theta r^2$

area of OAB = $\left(\frac{\pi - \theta}{2}\right)\left(\frac{1}{2}\right) r^2$ Apply formula with $\phi = \frac{\pi - \theta}{2}$,

$= \frac{(\pi - \theta) r^2}{4}$ $R = r$

substitute these into formula for area of logo

Area of logo = $2\theta r^2 + 2 \times \frac{(\pi - \theta) r^2}{4}$

$= 2\theta r^2 + \frac{(\pi - \theta) r^2}{2}$

Question 6 continued

$= \frac{1}{2}r^2 (4\theta + (\pi - \theta))$ Factor out $\frac{1}{2}r^2$, to rearrange to given solution.

$= \frac{1}{2}r^2 (3\theta + \pi)$

As required.

(c) Perimeter of logo = length OA + arc AB + length BC + arc CD + length DE + arc EF + length FO

Again, by congruence arc AB = arc EF

Remember:

$$\text{length of arc} = \frac{\underset{\text{angle}}{\phi}}{2\pi} \times \pi \times \underset{\text{radius}}{2R} = R\phi$$

arc CD $= 2r\theta$ substitute $R = 2r$, $\phi = \theta$

arc AB $= \left(\frac{\pi - \theta}{2}\right) \times r$ substitute $R = r$, $\phi = \frac{\pi - \theta}{2}$.

$AO = BC = DE = FO = r$

So, perimeter of logo =

$2r\theta + 2 \times (\pi - \theta)\frac{r}{2} + 4r$

$= 2r\theta + (\pi - \theta)r + 4r$ simplify

Question 6 continued

$= 2r\theta + \pi r - \theta r + 4r$

$= (4 + \pi + \theta)r.$

(Total for Question 6 is 5 marks)

7. **In this question you should show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

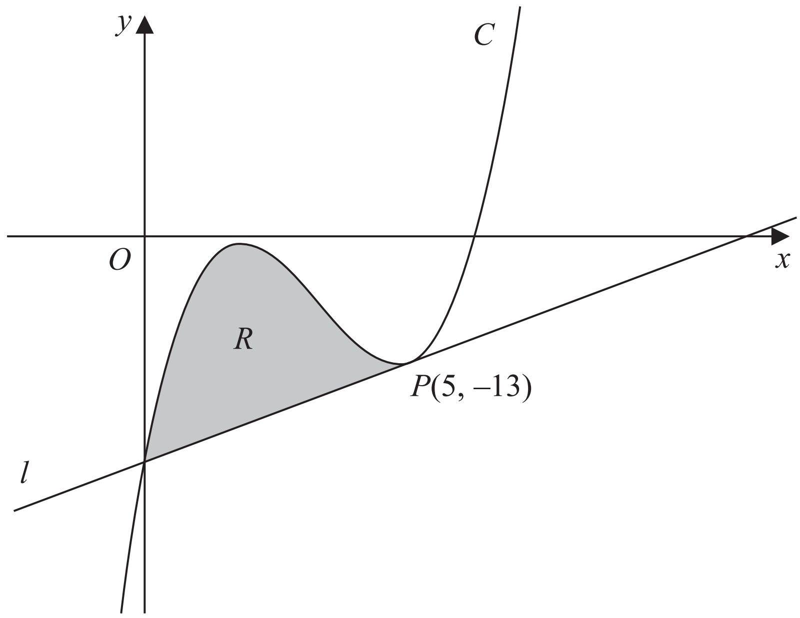

Figure 2

Figure 2 shows a sketch of part of the curve C with equation

$$y = x^3 - 10x^2 + 27x - 23$$

The point $P(5, -13)$ lies on C

The line l is the tangent to C at P

(a) Use differentiation to find the equation of l, giving your answer in the form $y = mx + c$ where m and c are integers to be found.

(4)

(b) Hence verify that l meets C again on the y-axis.

(1)

The finite region R, shown shaded in Figure 2, is bounded by the curve C and the line l.

(c) Use algebraic integration to find the exact area of R.

(4)

(a) l is tangent to c so gradient of l is $\frac{dy}{dx}$, evaluated at P.

$$\frac{dy}{dx} = 3x^2 - 20x + 27$$

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Question 7 continued

$$\left.\frac{dy}{dx}\right|_{x=5} = 3\times 5^2 - 20\times 5 + 27$$

$$= 3\times 25 - 100 + 27$$

$$= 75 - 100 + 27$$

$$= 102 - 100 \quad = 2$$

So gradient of l is 2.

Remember: Equation of line:

$y - y_1 = m(x - x_1)$ where (x_1, y_1) is point on line.

P is on l so substitute into equation of line.

$y - -13 = 2(x-5)$ ← Be careful not to mix up the x and y coords here!

$y + 13 = 2x - 10$

$y = 2x - 23$, $m = 2$, $c = -23$ Rearrange to required form.

You can check this is a sensible answer by looking at the graph and checking if gradient + y-intercept seem to agree with your values.

(b) From the equation for l, the y-intercept is $(0, -23)$ as $c = -23$.
So substitute $x = 0$ into the equation for C and check that $y = -23$

C: $y = 0^3 - 10\times 0^2 + 27\times 0 - 23$

Question 7 continued

$\Rightarrow \quad y = -23$

so l meets C again on the y-axis.

(c) Looking at the graph, we see that R is bounded by l at $x=0$ and $x=5$.
use integration to work out R.

R is the difference between C and l

$$R = \int_0^5 \underbrace{x^3 - 10x^2 + 27x - 23}_{C} - \underbrace{(2x - 23)}_{l}\, dx$$

$$= \int_0^5 x^3 - 10x^2 + 25x \, dx$$

$$= \left[\frac{1}{4}x^4 - \frac{10}{3}x^3 + \frac{25}{2}x^2\right]_0^5$$

$$= \frac{625}{4} - \frac{10}{3}(125) + \frac{25}{2}(25) - \left(\frac{1}{4}\times 0 - \frac{10}{3}\times 0 + \frac{25}{2}\times 0\right)$$

$$= \frac{625}{4} - \frac{1250}{3} + \frac{625}{2}$$

$$= \frac{625}{12}$$

8. The curve C has equation

$$px^3 + qxy + 3y^2 = 26$$

where p and q are constants.

(a) Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{apx^2 + bqy}{qx + cy}$$

where a, b and c are integers to be found.

(4)

Given that

- the point $P(-1, -4)$ lies on C
- the normal to C at P has equation $19x + 26y + 123 = 0$

(b) find the value of p and the value of q.

(5)

(a) This requires implicit differentiation.

$$px^3 + qxy + 3y^2 = 26$$

$$\frac{d}{dx}(px^3) + \boxed{\frac{d}{dx}(qxy)} + \boxed{\frac{d}{dx}(3y^2)} = \frac{d}{dx}(26)$$

product rule: $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$

$$\frac{d}{dx}(qx \cdot y) = y\frac{d}{dx}(qx) + qx\frac{d}{dx}(y)$$

$$= y \cdot q + qx \cdot \frac{dy}{dx} = qy + qx\frac{dy}{dx}$$

chain rule: $\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx}$

$$\frac{d}{dx}(3y^2) = \frac{d}{dy}(3y^2) \times \frac{dy}{dx} = 6y\frac{dy}{dx}$$

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Question 8 continued

so,

$$3px^2 + qy + qx\frac{dy}{dx} + 6y\frac{dy}{dx} = 0$$

$$3px^2 + qy + (6y + qx)\frac{dy}{dx} = 0$$

$$(6y + qx)\frac{dy}{dx} = -3px^2 - qy$$

Rearrange to required form

$$\frac{dy}{dx} = \frac{-3px^2 - qy}{qx + 6y}$$

so, $a = -3$, $b = -1$, $c = 6$.

(b) $P(-1, -4)$ lies on C so sub $x = -1$, $y = -4$ into eqn for C:

$$p(-1)^3 + q(-1)(-4) + 3(-4)^2 = 26$$

$$-p + 4q + 3 \times 16 = 26$$

$$-p + 4q = -22 \quad (1)$$

Equation of normal at P is $19x + 26y + 123 = 0$

Rearrange to get: $26y = -19x - 123$

$$y = -\frac{19}{26}x - 123$$

so gradient of normal $= -\frac{19}{26}$

Question 8 continued

Gradient of normal is negative reciprocal of gradient of curve, so m of normal × m of curve = −1.

so gradient of curve at P(−1, −4) = $\frac{26}{19}$.

so $\frac{26}{19} = \left.\frac{dy}{dx}\right|_{(-1,-4)}$

$$\frac{26}{19} = \frac{-3p(-1)^2 - q(-4)}{q(-1) + 6(-4)}$$

sub (−1, −4) into $\frac{dy}{dx}$ found earlier

$$\frac{26}{19} = \frac{-3p + 4q}{-q - 24}$$

$$-26q - 624 = -57p + 76q$$

$$-624 = -57p + 102q \quad (2)$$

we now have two simultaneous equations:
(you can also use your calculator to solve them)

(1) $-p + 4q = -22$

(2) $-57p + 102q = -624$

multiply (1) so the p coefficients will cancel

(1) × 57 : $-57p + 228 = -1254$

$-57p - -57p + 228q - 102q = -1254 - -624$

(2) − (1) × 57 : $0p + 126q = -630$

$$q = \frac{-630}{126} = -5$$

(1) $-p + 4 \times -5 = -22$

sub this q into (1) to find p.

Question 8 continued

$-p - 20 = -22$

$-p = -2$ Rearrange to solve for p.

$p = 2$

so, $p = 2$, $q = -5$

TIP: check your solutions by substituting your values back into your original equations.

(Total for Question 8 is 9 marks)

9. Show that

$$\sum_{n=2}^{\infty}\left(\frac{3}{4}\right)^{n}\cos(180n)^{\circ}=\frac{9}{28}$$

(3)

$$\sum_{n=2}^{\infty}\left(\frac{3}{4}\right)^{n}\cos(180n)^{\circ}$$

Expand sum to understand the pattern

$$= \left(\frac{3}{4}\right)^{2}\cos(360) + \left(\frac{3}{4}\right)^{3}\cos(540) + \left(\frac{3}{4}\right)^{4}\cos(720) + \dots$$

$$= \left(\frac{3}{4}\right)^{2}\times 1 + \left(\frac{3}{4}\right)^{3}\times -1 + \left(\frac{3}{4}\right)^{4}\times 1 + \dots$$

these cos values will alternate between 1 and -1 forever.

$$= \left(\frac{3}{4}\right)^{2} - \left(\frac{3}{4}\right)^{3} + \left(\frac{3}{4}\right)^{4} - \dots$$

$$= (-1)^{2}\left(\frac{3}{4}\right)^{2} + (-1)^{3}\left(\frac{3}{4}\right)^{3} + (-1)^{4}\left(\frac{3}{4}\right)^{4} + \dots$$

$$= \left(-\frac{3}{4}\right)^{2} + \left(-\frac{3}{4}\right)^{3} + \left(-\frac{3}{4}\right)^{4} + \dots$$

This is now the sum of a geometric sequence.

Formula:

first term

$$S_{\infty} = \frac{a}{1-r}$$

common ratio.

Sum starts at $n=2$, so first term:

$$a = \left(\frac{3}{4}\right)^{2}\cos(360) = \frac{9}{16}\times 1 = \frac{9}{16}.$$

Question 9 continued

common ratio $= -\frac{3}{4}$.

So $\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n \cos(180n) = \frac{a}{1-r}$ substitute $a = \frac{9}{16}$, $r = -\frac{3}{4}$

$= \frac{\frac{9}{16}}{1-\left(-\frac{3}{4}\right)} = \frac{\frac{9}{16}}{\frac{7}{4}}$ $\times 16$

$= \frac{9}{28}$. As required.

(Total for Question 9 is 3 marks)

10. The time, T seconds, that a pendulum takes to complete one swing is modelled by the formula

$$T = al^b$$

where l metres is the length of the pendulum and a and b are constants.

(a) Show that this relationship can be written in the form

$$\log_{10} T = b\log_{10} l + \log_{10} a$$

(2)

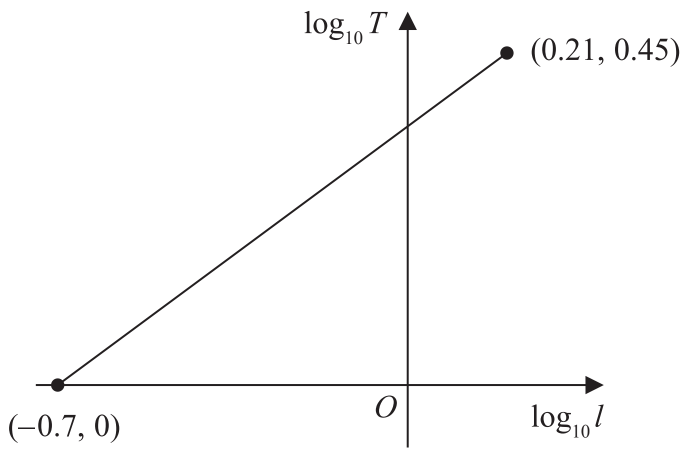

Figure 3

A student carried out an experiment to find the values of the constants a and b.

The student recorded the value of T for different values of l.

Figure 3 shows the linear relationship between $\log_{10} l$ and $\log_{10} T$ for the student's data. The straight line passes through the points (–0.7, 0) and (0.21, 0.45)

Using this information,

(b) find a complete equation for the model in the form

$$T = al^b$$

giving the value of a and the value of b, each to 3 significant figures.

(3)

(c) With reference to the model, interpret the value of the constant a.

(1)

(a) start with formula given:

$T = al^b$

$\log_{10} T = \log_{10}(al^b)$

Apply $\log_{10}$ to both sides to get the LHS required.

Question 10 continued

$\log_{10} T = \log_{10} a + \log_{10}(L^b)$ log law: $\log xy = \log x + \log y$

$= \log_{10} a + b\log_{10} L$ log law: $\log x^y = y\log x$

$= b\log_{10} L + \log_{10} a$ Rearrange to form given.

(b) use the information on the graph to find the equation of the line.

$m = \dfrac{0.45 - 0}{0.21 - (-0.7)}$ use $m = \dfrac{y_1 - y_2}{x_1 - x_2}$ with $(x_1, y_1) = (0.21, 0.45)$, $(x_2, y_2) = (-0.7, 0)$

$= \dfrac{0.45}{0.91}$

$= \dfrac{45}{91}$

so $y = \dfrac{45}{91}x + c$.

use $(-0.7, 0)$ and substitute $x = -0.7$, $y = 0$ to find c.

$0 = \dfrac{45}{91} \times -0.7 + c$ Rearrange to find c.

$\Rightarrow c = \dfrac{9}{26}$

so, equation of line : $y = \dfrac{45}{91}x + \dfrac{9}{26}$

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Question 10 continued

But the graph is labelled such that
$y = \log_{10} T$, $x = \log_{10} l$

so substitute these in the line equation

$$\log_{10} T = \frac{45}{91} \log_{10} l + \frac{9}{26}$$

compare this to (a): $\log_{10} T = b \log_{10} l + \log_{10} a$

and note that:

$b = \frac{45}{91}$, $\log_{10} a = \frac{9}{26}$

$= 0.495$ $a = 10^{\frac{9}{26}} = 2.22$ (3sf)

so complete equation : $T = \underbrace{0.495}_{b} \times l^{\overset{2.22}{a}}$

(c) consider when $l = 1$.

$T = a l^{b} = a \times 1^{b} = a \times 1 = a$

so a is the time taken for a pendulum of length 1m to complete one swing.

11.

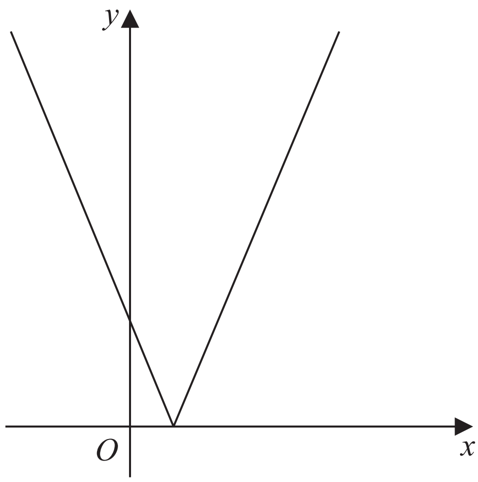

Figure 4

Figure 4 shows a sketch of the graph with equation

$$y = |2x - 3k|$$

where k is a positive constant.

(a) Sketch the graph with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = k - |2x - 3k|$$

stating

- the coordinates of the maximum point
- the coordinates of any points where the graph cuts the coordinate axes

(4)

(b) Find, in terms of k, the set of values of x for which

$$k - |2x - 3k| > x - k$$

giving your answer in set notation.

(4)

(c) Find, in terms of k, the coordinates of the minimum point of the graph with equation

$$y = 3 - 5\mathrm{f}\left(\frac{1}{2}x\right)$$

(2)

(a) $y = f(x) = k - |2x - 3k|$

taking away a positive

absolute value so always ≥ 0.

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Question 11 continued

Therefore the max f(x) will occur when $|2x-3k|$ is at it's smallest.

smallest non-negative value is 0.
so max f(x) when $|2x-3k| = 0$.

$$|2x-3k| = 0$$

solve $|2x-3k|=0$ to find x- value of max point.

$$\Rightarrow \quad 2x = 3k$$

$$x = \frac{3k}{2}$$

$y = f(x) = k - |2x-3k|$

so y-value at max point:

$$y = k - 0 = k$$

so coordinates of max point: $\left(\frac{3}{2}k, k\right)$

Note that k is positive.

Graph cuts axes when $x=0$ and when $y=0$.

x- intercept: $0 = k - |2x-3k|$

sub $y=0$ into $y=f(x)$.

$$|2x-3k| = k$$

square to get rid of modulus.

$$\Rightarrow \quad (2x-3k)^2 = k^2$$

$$\Rightarrow 4x^2 - 12kx + 9k^2 = k^2$$

	$2x$	$-3k$
$2x$	$4x^2$	$-6kx$
$-3k$	$-6kx$	$9k^2$

Question 11 continued

$\Rightarrow 4x^2 - 12kx + 8k^2 = 0$

use $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \frac{12k \pm \sqrt{144k^2 - 4 \times 4 \times 8k^2}}{2 \times 4}$

with $a = 4$
$b = -12k$
$c = 8k^2$.

$= \frac{12k \pm \sqrt{16k^2}}{8}$

Simplify

$= \frac{12k \pm 4k}{8}$

$= \frac{3k \pm k}{2}$

split into two separate solutions.

so $x = \frac{3k + k}{2}$, $x = \frac{3k - k}{2}$

$x = 2k$, $x = k$

so, x-intercepts at $(k, 0)$, $(2k, 0)$

y-intercepts: $y = k - |0 - 3k|$

sub $x=0$ into $y = f(x)$

$y = k - |-3k|$

Absolute value of $-3k$ is $3k$

$y = k - (3k)$

$= -2k$

so y-intercept : $(0, -2k)$

max point: $\left(\frac{3}{2}k, k\right)$. x-intercepts: $(k, 0)$, $(2k, 0)$,
y-intercept: $(0, -2k)$

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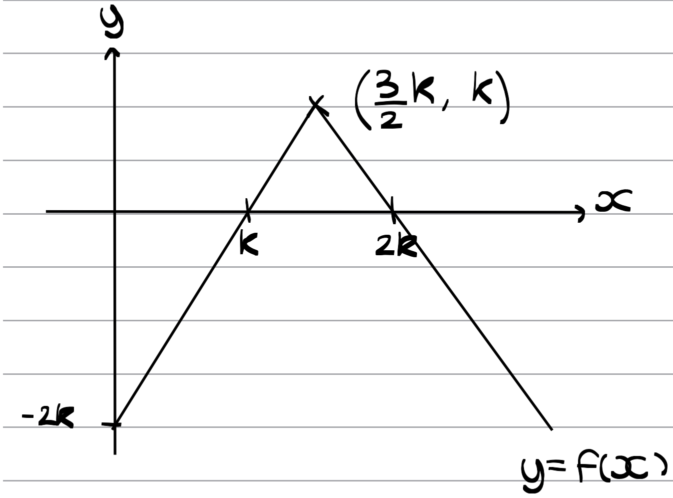
Question 11 continued

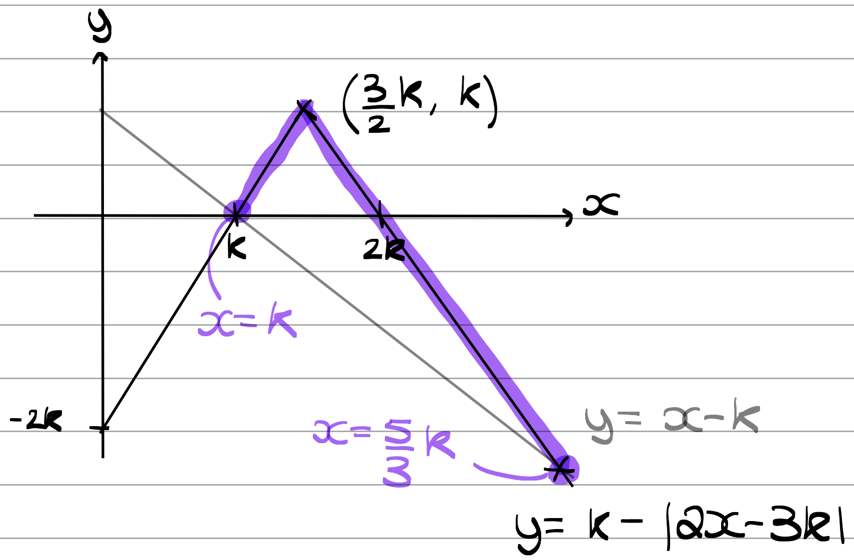


• start by plotting max point + intercepts.

• note that it will have a similar shape to the $y = |2x - 3k|$ graph given in question

TIP: You can use a graphical calculator to check the shape of your graph by assigning k a value and plotting that.

(b) consider the problem graphically.



To work out intercepts, consider when $k - |2x - 3k| = x - k$

$$k - |2x - 3k| = x - k$$

(Total for Question 11 is 10 marks)

Q11 (b) cont.

$$\Rightarrow \quad -|2x-3k| = x-2k$$

square to eliminate modulus.

$$\Rightarrow \quad (2x-3k)^2 = (x-2k)^2$$

$$\Rightarrow \quad 4x^2 - 12kx + 9k^2 = x^2 - 4kx + 4k^2$$

$$\Rightarrow \quad 3x^2 - 8kx + 5k^2 = 0$$

Solve quadratic:

method 1: factorise

	$3x$	$-5k$
x	$3x^2$	$-5kx$
$-k$	$-3kx$	$5k^2$

$$\Rightarrow \quad (3x-5k)(x-k) = 0$$

$$\Rightarrow \quad x = \frac{5}{3}k, \quad x = k$$

method 2: quadratic formula

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$a = 3, \quad b = -8k, \quad c = 5k^2$

$$\Rightarrow x = \frac{8k \pm \sqrt{64k^2 - 4\times3\times5k^2}}{2\times3}$$

$$= \frac{8k \pm \sqrt{4k^2}}{6}$$

$$= \frac{8k \pm 2k}{6} = \frac{4k \pm k}{3}$$

$$\Rightarrow \quad x = \frac{4k+k}{3} = \frac{5}{3}k$$

$$\text{or} \quad x = \frac{4k-k}{3} = k$$

Add these intercepts to graph.

From looking at graph, inequality is satisfied when

$$k < x < \frac{5}{3}k$$

Strictly less than, same as in question.

Q11 (c). $y = 3 - 5f(\frac{1}{2}x)$

This is a transformation of $y = f(x)$:

Transformations:

$y = f(x)$
↓ horizontal stretch, s.f. 2
$y = f(\frac{1}{2}x)$
↓ vertical stretch s.f. 5
$y = 5f(\frac{1}{2}x)$
↓ reflection in $y=0$
$y = -5f(\frac{1}{2}x)$
↓ vertical shift up 3.
$y = 3 - 5f(\frac{1}{2}x)$

To find min point of transformed graph, apply these transformations to the max point of $y=f(x)$: $(\frac{3}{2}k, k)$

$(\frac{3}{2}k, k)$
↓ horizontal stretch s.f 2.
$(3k, k)$
↓ vertical stretch s.f. 5
$(3k, 5k)$
↓ reflection in $y=0$
$(3k, -5k)$
↓ vertical shift up 3.
$(3k, 3-5k)$

max point becomes min because it is reflected in the x-axis.

so min point : $(3k, 3-5k)$

12. (a) Use the substitution $u = 1 + \sqrt{x}$ to show that

$$\int_0^{16} \frac{x}{1+\sqrt{x}}\, dx = \int_p^q \frac{2(u-1)^3}{u}\, du$$

where p and q are constants to be found.

(3)

(b) Hence show that

$$\int_0^{16} \frac{x}{1+\sqrt{x}}\, dx = A - B \ln 5$$

where A and B are constants to be found.

(4)

(a) $u = 1 + \sqrt{x} = 1 + x^{1/2}$

use substitution $u = 1+\sqrt{x}$ to find x in terms of u.

$\sqrt{x} = u - 1$

$\boxed{x = (u-1)^2}$

$\frac{du}{dx} = \frac{1}{2} x^{-1/2} = \frac{1}{2} \cdot \frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}$

Find $\frac{du}{dx}$

$du = \frac{1}{2\sqrt{x}}\, dx$

Rearrange to find dx in terms of x and du.

$dx = 2\sqrt{x}\, du$

use $\sqrt{x} = u-1$, as earlier to find dx in terms of u, du.

$\boxed{dx = 2(u-1)\, du}$

Substitute these into the integral to find it in terms of u and du.

$(u-1)^2$

$$\int_0^{16} \frac{x}{1+\sqrt{x}}\, dx = \int_{x=0}^{x=16} \frac{(u-1)^2}{u} \cdot 2(u-1)\, du$$

u $\quad 2(u-1)du$

Question 12 continued

$$= \int_{x=0}^{x=16} \frac{2(u-1)^3}{u}\, du.$$ Bounds still in terms of x.

Use $x = (u-1)^2$ to find bounds in terms of u.

$0 = (u-1)^2$

$u-1 = 0 \Rightarrow u = 1$

$16 = (u-1)^2$

$u-1 = \pm 4$ take positive here, as upper bound must be greater than lower bound, 1.

$u-1 = 4 \Rightarrow u = 5$

Hence, $$\int_0^{16} \frac{x}{1+\sqrt{x}}\, dx = \int_1^5 \frac{2(u-1)^3}{u}\, du.$$ As required.

$p = 1, \quad q = 5$ Check answer is in required form.

(b) Now solve the integral in terms of u.

$$\int_1^5 \frac{2(u-1)^3}{u}\, du$$ multiply out $(u-1)^3$ bracket using Bernoulli pyramid

$$= \int_1^5 \frac{2(u^3 - 3u^2 + 3u - 1)}{u}\, du$$

1
1 1
1 2 1
→ 1 3 3 1 ←
1 4 6 4 1

or $(u^2 - 2u + 1)(u-1)$.

Question 12 continued

$$= \int_1^5 \frac{2u^3 - 6u^2 + 6u - 2}{u} \, du$$

divide through by u.

$$= \int_1^5 2u^2 - 6u + 6 - \frac{2}{u} \, du$$

$$= \left[\frac{2}{3}u^3 - \frac{6}{2}u^2 + 6u - 2\ln u\right]_1^5$$

Remember: $\int \frac{1}{x} dx = \ln x + c$

$$= \frac{2}{3} \times 5^3 - \frac{6}{2} \times 5^2 + 6 \times 5 - 2\ln 5 - \left(\frac{2}{3} \times 1 - \frac{6}{2} \times 1 + 6 \times 1 - 2\ln 1\right)$$

sub numbers and take bottom from top.

$$= \frac{250}{3} - 75 + 30 - 2\ln 5 - \left(\frac{2}{3} - 3 + 6 - 0\right)$$

$$= \frac{250}{3} - 75 + 30 - 2\ln 5 - \frac{2}{3} + 3 - 6$$

$$= \frac{104}{3} - 2\ln 5$$

compare to form in question.

so $A = \frac{104}{3}$, $B = -2$

You can check your integration is correct on your calculator.

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13. The curve C has parametric equations

$$x = \sin 2\theta \qquad y = \operatorname{cosec}^3 \theta \qquad 0 < \theta < \frac{\pi}{2}$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of θ

(3)

(b) Hence find the exact value of the gradient of the tangent to C at the point where $y = 8$

(3)

(a) First remember $\operatorname{cosec} x = \frac{1}{\sin x}$

so $y = \left(\frac{1}{\sin\theta}\right)^3 = (\sin\theta)^{-3}$

use chain rule: $\frac{dy}{dx} = \frac{dy}{d\theta} \times \underbrace{\frac{d\theta}{dx}}_{= 1 \div \frac{dx}{d\theta}}$.

$\frac{dy}{d\theta} = \frac{d}{d\theta}((\sin\theta)^{-3})$

sub $u = \sin\theta$

$\frac{du}{d\theta} = \cos\theta$

use chain rule: $\frac{dy}{d\theta} = \frac{dy}{du} \times \frac{du}{d\theta}$

$\frac{dy}{d\theta} = \frac{d}{d\theta}(u^{-3}) = \frac{d}{du}(u^{-3}) \times \cos\theta$

$\frac{dy}{d\theta} = -3u^{-4} \times \cos\theta$

$\boxed{\frac{dy}{d\theta} = -3\cos\theta \sin^{-4}\theta}$.

$\frac{dx}{d\theta} = \frac{d}{d\theta}(\sin 2\theta) = 2\cos 2\theta$ using chain rule again.

P 6 8 7 3 2 A 0 3 8 4 8

Question 13 continued

so $\boxed{1 \div \frac{dx}{d\theta} = \frac{1}{2\cos 2\theta}}$

sub $\frac{dy}{d\theta}$, $\frac{dx}{d\theta}$ into $\frac{dy}{dx}$.

so $\frac{dy}{dx} = \frac{dy}{d\theta} \times \left(1 \div \frac{dx}{d\theta}\right)$

$= -\frac{3\cos\theta}{\sin^4\theta} \times \frac{1}{2\cos 2\theta}$

$= \frac{-3\cos\theta}{2\sin^4\theta\cos 2\theta}$

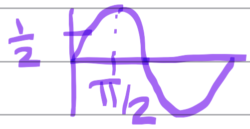
(b) on C, when $y = 8$,

$8 = \left(\frac{1}{\sin\theta}\right)^3 \Rightarrow 2 = \frac{1}{\sin\theta}$

$\Rightarrow \sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

verify only one solution in $(0, \frac{\pi}{2})$ using sin graph



sub $\theta = \pi/6$ into $\frac{dy}{dx}$ to find gradient

$\frac{dy}{dx} = \frac{-3\cos(\pi/6)}{2\sin^4(\pi/6)\cos(2\pi/6)}$

(Total for Question 13 is 6 marks)

Q13(b) cont.

$$\frac{dy}{dx} = \frac{-3 \times \frac{\sqrt{3}}{2}}{2 \times \left(\frac{1}{2}\right)^4 \times \frac{1}{2}} = \frac{-3 \times \frac{\sqrt{3}}{2}}{\frac{1}{16}} = \underline{-24\sqrt{3}}$$

asked for exact value so leave in surd.

14.

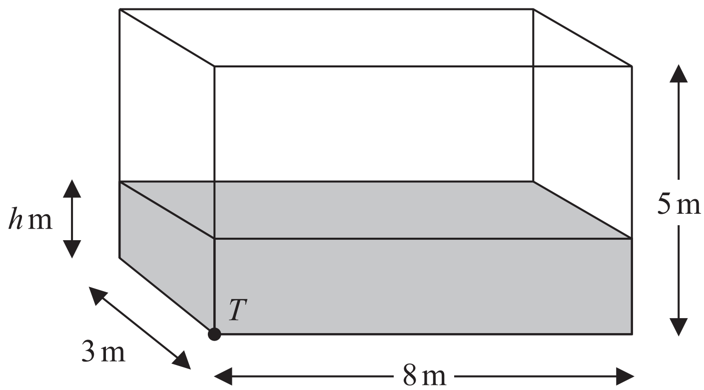


Figure 5

Water flows at a constant rate into a large tank.

The tank is a cuboid, with all sides of negligible thickness.

The base of the tank measures 8 m by 3 m and the height of the tank is 5 m.

There is a tap at a point T at the bottom of the tank, as shown in Figure 5.

At time t minutes after the tap has been opened

- the depth of water in the tank is h metres
- water is flowing into the tank at a constant rate of $0.48\,\text{m}^3$ per minute
- water is modelled as leaving the tank through the tap at a rate of $0.1h\,\text{m}^3$ per minute

(a) Show that, according to the model,

$$1200\frac{dh}{dt} = 24 - 5h$$

(4)

Given that when the tap was opened, the depth of water in the tank was 2 m,

(b) show that, according to the model,

$$h = A + Be^{-kt}$$

where A, B and k are constants to be found.

(6)

Given that the tap remains open,

(c) determine, according to the model, whether the tank will ever become full, giving a reason for your answer.

(2)

(a) Depth of water is h so rate of change is $\frac{dh}{dt}$.

Question 14 continued

Rate of flow measures volume of water, not height of water level.
so work out rate in terms of height.

Tank has base of $8 \times 3 = 24m^2$.

so in order to increase height by 1cm, must increase volume by $24m^3$.

so, rate in $= 0.48m^3 / min$

so per minute, height increases by $\frac{0.48}{24} m^2$.

so height rate increase $= 0.02m^2 / min$.

Repeat for rate out:

rate out $= 0.1h \, m^3 / min$

so rate of height decrease $= \frac{0.1h}{24} = \frac{h}{240}$

So combine these to create formula for overall rate of change.

$$\frac{dh}{dt} = 0.02 - \frac{h}{240}$$

total rate of change = rate of increase - rate of decrease.

$$240 \frac{dh}{dt} = \frac{24}{5} - h$$

multiply through to get whole numbers.

$$1200 \frac{dh}{dt} = 24 - 5h,$$

as required.

Question 14 continued

(b) $1200\frac{dh}{dt} = 24 - 5h$

$$\frac{1}{24-5h}\frac{dh}{dt} = \frac{1}{1200}$$

$$\int \frac{1}{24-5h}\,dh = \int \frac{1}{1200}\,dt$$

integrate by inspection or by substitution:

$u = 24 - 5h$

$\frac{du}{dh} = -5 \Rightarrow dh = -\frac{1}{5}du$

$$\int \frac{1}{24-5h}\,dh = \int -\frac{1}{5}\times\frac{1}{u}\,du = -\frac{1}{5}\ln u + c = -\frac{1}{5}\ln(24-5h) + c$$

$$-\frac{1}{5}\ln(24-5h) = \frac{1}{1200}t + c$$

collect both constants of integration on RHS

$$\ln(24-5h) = -\frac{1}{240}t + c$$

$e^{-\frac{1}{240}t + c} = e^{-\frac{1}{240}t} \times \underbrace{e^{c}}_{\text{constant} = K}$

$$24 - 5h = Ke^{-\frac{1}{240}t}$$

$$5h = 24 - Ke^{-\frac{1}{240}t}$$

$$h = \frac{24}{5} - \frac{K}{5}e^{-\frac{t}{240}}$$

consider, initially depth of water was 2m so substitute $t=0$, $h=2$.

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Question 14 continued

$2 = \frac{24}{5} - \frac{K}{5} \times e^{0}$ $e^{0} = 1$

$2 = \frac{24}{5} - \frac{K}{5}$ Rearrange for B.

$10 = 24 - K$

$K = 14.$

so $h = \frac{24}{5} - \frac{14}{5} e^{\frac{1}{240}t}$

$A = \frac{24}{5}$, $B = -\frac{14}{5}$, $K = \frac{1}{240}$

(c) Tank becomes full when h= 5m.
calculate t when h = 5.

$5 = \frac{24}{5} - \frac{14}{5} e^{\frac{t}{240}}$

$\frac{1}{5} = -\frac{14}{5} e^{\frac{t}{250}}$ Rearrange for t.

$\Rightarrow \frac{t}{250} = \ln\left(-\frac{1}{14}\right)$ cannot take ln of a negative

No solutions for t.

so no times when tank is full.

so No, the tank will never become full.

(Total for Question 14 is 12 marks)

15. (a) Express $2\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \dfrac{\pi}{2}$

Give the exact value of R and the value of α in radians to 3 decimal places.

(3)

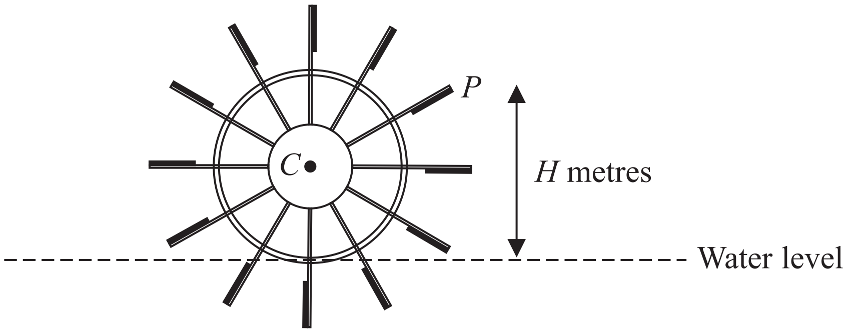


Figure 6

Figure 6 shows the cross-section of a water wheel.

The wheel is free to rotate about a fixed axis through the point C.

The point P is at the end of one of the paddles of the wheel, as shown in Figure 6.

The water level is assumed to be horizontal and of constant height.

The vertical height, H metres, of P above the water level is modelled by the equation

$$H = 3 + 4\cos(0.5t) - 2\sin(0.5t)$$

where t is the time in seconds after the wheel starts rotating.

Using the model, find

(b) (i) the maximum height of P above the water level,

(ii) the value of t when this maximum height first occurs, giving your answer to one decimal place.

(3)

In a single revolution of the wheel, P is below the water level for a total of T seconds.

According to the model,

(c) find the value of T giving your answer to 3 significant figures.

(*Solutions based entirely on calculator technology are not acceptable*.)

(4)

In reality, the water level may not be of constant height.

(d) Explain how the equation of the model should be refined to take this into account.

(1)

Question 15 continued

(a) $2\cos\theta - \sin\theta = R\cos(\theta + a)$

Remember: $\cos(A+B) = \cos A\cos B - \sin A\sin B$

so $R\cos(\theta + a) = R\cos\theta\cos a - R\sin\theta\sin a$

So $2\cos\theta - \sin\theta = R\cos a\cos\theta - R\sin a\sin\theta$

Now match up: $2 = R\cos a$ (1)

$1 = R\sin a$ (2)

Now we have 2 equations in 2 variables (R and a), so we can solve them.

$1 = R\sin a$

$2 = R\cos a$

$\frac{1}{2} = \frac{\cancel{R}\sin a}{\cancel{R}\cos a} = \tan a$

$a = \arctan\left(\frac{1}{2}\right) = \underline{0.464}$

question wants 3dp

check you are using radians!

Now, find R using $R^2 = 1^2 + 2^2$.

Two ways of thinking of this:

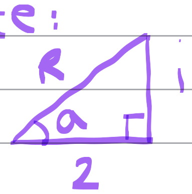
$1^2 = R^2\sin^2 a$

$+\ 2^2 = R^2\cos^2 a$

$1^2 + 2^2 = R^2(\underbrace{\sin^2 a + \cos^2 a}_{=1})$

$1^2 + 2^2 = R^2$.

using SOHCAHTOA, create:



By Pythagoras, $R^2 = 1^2 + 2^2$.

Question 15 continued

so, $R^2 = 1^2 + 2^2$

$= 1 + 4$

$R^2 = 5 \Rightarrow R = \sqrt{5}$

so $2\cos\theta - \sin\theta = \sqrt{5}\cos(\theta + 0.464)$

(b) $H = 3 + 4\cos(0.5t) - 2\sin(0.5t)$

$= 3 + 2(\underline{2\cos(0.5t) - \sin(0.5t)})$

By taking out a factor of 2, this looks like (a).
so let $\theta = 0.5t$ and substitute solution to (a)

$H = 3 + 2\sqrt{5}\cos(0.5t + 0.464)$

(i) max value cos can take is 1.
so max height when cos term is 1.

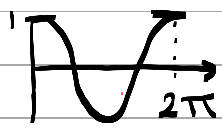
max $H = 3 + 2\sqrt{5} \times 1$

max $H = 3 + 2\sqrt{5}$.

(ii) max height when cos term is 1.

$1 = \cos(0.5t + 0.464)$ solve for t.

$0.5t + 0.464 = \arccos(1) = 0, 2\pi$



$0.5t = -0.464$ or $0.5t = 2\pi - 0.464$

$t = -0.928$ $\quad$ $\underline{t = 11.6}$ 1dp

t is a time so cannot be negative, so max height first occurs when $t = 11.6$.

Question 15 continued

(c) P is below the water level when H is negative.

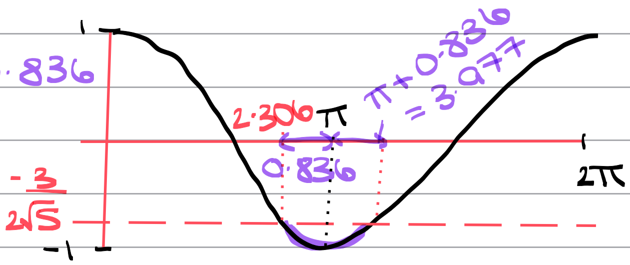
$$H < 0$$

$$\Rightarrow \quad 3 + 2\sqrt{5}\cos(0.5t + 0.464) < 0$$

$$\Rightarrow \quad \cos(0.5t + 0.464) < -\frac{3}{2\sqrt{5}}$$

$$\Rightarrow \quad 0.5t + 0.464 < \arccos\left(-\frac{3}{2\sqrt{5}}\right)$$

$$\arccos\left(\frac{-3}{2\sqrt{5}}\right) = 2.306,\ 3.977$$



use symmetry of cos graph to find other solution.

Looking at graph, we can see its less than $\arccos\left(\frac{-3}{2\sqrt{5}}\right)$ between 2.306 and 3.977

so $\quad 2.306 < 0.5t + 0.464 < 3.977$

below water when strictly less than 0.

$$1.842 < 0.5t < 3.513$$

$$3.684 < t < 7.026$$

Rearrange on both sides of inequality.

so $T = 7.026 - 3.684 = 3.342 \simeq 3.34$ (3sf)

Question 15 continued

(d) consider $H = 3 + \underbrace{4\cos(0.5t) - 2\sin(0.5t)}_{\text{only dependant on time}}$

so don't want to change this part of the model

so to refine model, should change this "3" value.

ie, could introduce a trig function to reflect the change in water level.

(Total for Question 15 is 11 marks)

TOTAL FOR PAPER IS 100 MARKS

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